Available online at www.sciencedirect.com
ScienceDirect
International Journal of Heat and Mass Transfer 51 (2008) 4583–4588
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Technical Note

A note on methods for analysis of flow through microchannels

G. Chakraborty *

Department of Mechanical Engineering, Indian Institute of Technology, Kharagpur 721 302, India

Received 30 May 2007; received in revised form 4 October 2007
Available online 20 April 2008

Abstract

The flow problem within a straight microchannel of arbitrary cross section is analyzed. Exact analytical solutions for flow profile of a channel flow with no-slip boundary conditions have been obtained in literature only for simple geometry of channel section. In this paper, a number of problems with more complicated geometries are solved either exactly or approximately. Three general solution methods are discussed, namely, complex function analysis, membrane vibration analogy and variational method. The usefulness of each method is justified with the help of examples.

1. Introduction

Microfluidics has become an active field of research following development of micro devices like micro sensors, micro mixers which find application in various fields of science and engineering [1]. The flow problem within a straight microchannel has been and still is a subject of research because in the micro level the flow shows significant deviation from that within a macrochannel. In the literature, the problem has been analytically solved only for a few simple cross sectional geometries. The geometry of a microchannel can, however, be complicated due to manufacturing restrictions. For example, the cross section may have Gaussian profile during laser ablation in the surface of polymer PMMA, the sidewalls of a rectangular channel may have wall slope etc. The aim of this paper is to describe various methods of analysis of fluid flow in a straight microchannel of arbitrary cross section. As will be shown, great many cases can be analyzed.

Three analytical methods are described in this paper. In the first method, functions of a complex variable are effectively used, whereas, the second method exploits the analogy of the problem with membrane vibration. In the third method, a variational formulation of the problem is given that can often aid in approximately calculating the velocity profile within a channel of arbitrary cross section.

In this paper, it is assumed that the pressure driven flow in microchannel is incompressible viscous flow governed by Navier–Stokes equation, where inertial forces can be neglected. No-slip boundary condition is assumed though for very narrow channels this boundary condition may not hold [2–4]. It is to be mentioned that although the analysis has been carried out for microchannels, same flow equations appear as well in macrochannels if the linear flow is assumed to be steady, fully developed and laminar [5].

2. Problem statement

Consider viscous incompressible flow within a straight microchannel of uniform cross section. The steady flow velocity $u(y, z)$ along axial direction is governed by the following equation

$$\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\mathrm{d}p}{\mathrm{d}x}, \tag{1}$$

where $\frac{\mathrm{d}p}{\mathrm{d}x}$ is the constant pressure gradient along the length axis x. The y- and z-axes are orthogonal to the length axis and x–y–z form a right handed co-ordinate system. No-slip condition at the boundary is given as

$$u(y,z)=0 \quad \text{at} \quad \varphi(y,z)=0, \tag{2}$$

* Tel.: +91 3222 282994.
E-mail address: goutam@mech.iitkgp.ernet.in

0017-9310/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2007.11.058

$$\text{or}\quad \frac{\partial u}{\partial s} = 0 \quad \text{along} \quad \varphi(y,z) = 0, \tag{3}$$

where s is the length measured along the boundary represented as $\varphi(y,z) = 0$. The boundary curve is assumed to be rectifiable.

So far, the exact velocity profile has been obtained for a limited number of cases, e.g., flow through circular or elliptical pipes, channel with rectangular cross section [6]. However, the form of the mathematical problem appears in solid mechanics, for example, the problem of torsion of a shaft with non-circular cross section [7], deflection of membrane under constant load. If, analogy with physical problems are considered then a number of problems can be solved either exactly or approximately. Three solution methods are discussed below.

3. Closed form solution using complex functions

In this section, an exact analytical solution technique using complex function analysis is given. The solution of Eq. (1) can be given as

$$u(y,z) = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)(y^2 + z^2) + u_1(y,z), \tag{4}$$

where the function $u_1(y,z)$ satisfies the following equation

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} = 0. \tag{5}$$

It is well known that complex functional analysis can be used to solve Eq. (5). In fact, the solution is either the real part or the imaginary part of an analytic function $f(y + \mathrm{i}z)$ where $\mathrm{i} = \sqrt{-1}$. If the function f is chosen in such a way that $u(y,z)$ vanishes at the boundary, i.e., Eqs. (2) or (3) is satisfied then the corresponding $u(y,z)$ is the solution of the original problem. The method is explained with a few examples.

Example 1. To find the velocity profile of flow within a channel of circular cross section whose boundary is given by $\varphi(y,z) = y^2 + z^2 - a^2 = 0$, the complex function $f(y + \mathrm{i}z)$ is taken as a constant, say C. If $C = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)a^2$, then the flow profile becomes

$$u(y,z) = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)(y^2 + z^2 - a^2) \tag{6}$$

or in polar coordinate $u(y,z) = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)(r^2 - a^2)$. This is the solution of the given problem as it satisfies the boundary condition.

Example 2. In this example the cross section is assumed to be in the form of an equilateral triangle. Let the sides of the triangle are represented as

$$y - \sqrt{3}z - \frac{2}{3}\alpha = 0, \tag{7}$$

$$y + \sqrt{3}z - \frac{2}{3}\alpha = 0 \tag{8}$$

and

$$y + \frac{1}{3}\alpha = 0. \tag{9}$$

In this problem assume $f(\xi) = a\xi^3 + b$ where $\xi = y + \mathrm{i}z$ with a, b as arbitrary constants. Taking the real part of f as $u_1(y,z)$, one gets,

$$u_1(y,z) = a(y^3 - 3yz^2) + b, \tag{10}$$

yielding

$$u(y,z) = \frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left[\frac{1}{2}(y^2 + z^2) - \frac{1}{2\alpha}(y^3 - 3yz^2) + \beta\right], \tag{11}$$

where $a = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\frac{1}{\alpha}$ and $b = \frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\beta$. For the given problem setting $\beta = -\frac{2}{27}\alpha^2$, one finally gets

$$u = -\frac{1}{4\mu}\frac{1}{\alpha}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(y - \sqrt{3}z - \frac{2}{3}\alpha\right)\left(y + \sqrt{3}z - \frac{2}{3}\alpha\right)\left(y + \frac{1}{3}\alpha\right). \tag{12}$$

Eq. (12) is seen to produce the exact solution of the problem as it vanishes at the boundaries represented by Eqs. (7)–(9).

Example 3. Consider a channel with cross section shown in Fig. 1. The section is made of two circular arcs, one of a circle of radius $b$with origin at center and the other of circle of radius a with origin at $(a, 0)$. In order to obtain flow profile the use of polar co-ordinates is most effective. The complex function is taken as $f(\xi) = A\xi + \frac{B}{\xi} + C$, where A, B and C are constants. Taking the real part of f one gets the solution of Eq. (5) as

$$u_1(r,\theta) = Ar\cos\theta + \frac{B}{r}\cos\theta + C, \tag{13}$$

where $r^2 = y^2 + z^2$ and $\tan\theta = z/y$. For the given problem choose $A = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right), B = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)2b^2a$ and $C = -\frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)b^2$. The flow profile then becomes

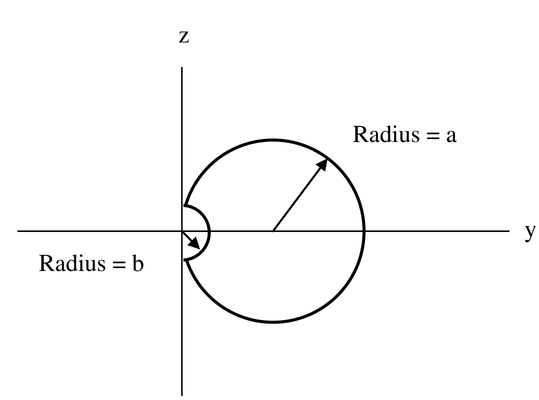

Fig. 1. Channel cross section made by two intersecting circles.

$$u = \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left[r^2 - b^2 - 2ar\cos\theta + \frac{2ab^2}{r}\cos\theta\right]$$
$$= \frac{1}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)(r^2 - b^2)\left(1 - \frac{2a}{r}\cos\theta\right) \quad (14)$$

which happens to be exact solution of the flow problem since it vanishes at the boundaries $r^2 - b^2 = 0$ and $r = 2a\cos\theta$.

It may be mentioned that similar mathematical problem arises in calculation of shear stress upon application of torque on a straight elastic shaft of arbitrary cross section. The solid mechanics problem, named after Saint-Venant [8], has been analyzed by various researchers. The solutions for different complicated cross sections have been obtained [9,10]. In microfluidics most of the problems are of theoretical interest only since the boundaries are too complicated to appear in reality.

From the examples given above it is seen that although the complex functional analysis is capable of providing exact solutions of many problems the efficacy of the method relies on suitable choice of the complex function $f(y + iz)$ for a given boundary. Considering this difficulty, the following method, inspired by the analogy of transverse vibration problem of a taut membrane is proposed.

4. Solution in infinite series form

In this section the analogy between the flow problem and the problem of vibration of a taut membrane is exploited to obtain solution in an infinite series form. Consider free transverse vibration of a membrane having uniform unit tension in all sides and unit thickness and material density. The equation of motion is given by [11]

$$\frac{\partial^2 w}{\partial t^2} - \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = 0, \quad (15)$$

where w is the transverse deflection of any point within the membrane. The deflection w vanishes at the boundary. For free vibration with frequency, ω, the response is assumed as $w(y,z,t) = W(y,z)\cos\omega t$ and substituted into Eq. (15) to get

$$\left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right) = -\omega^2 W(y,z). \quad (16)$$

The solution together with the boundary conditions yield different mode shapes. The normal modes possess orthogonal property [11], that is, for the mth and nth mode shapes $W_m(y,z)$ and $W_n(y,z)$, the following relationship exists:

$$\int W_m(y,z)W_n(y,z)\,\mathrm{d}y\,\mathrm{d}z = 0 \quad \text{if} \quad m \neq n. \quad (17)$$

Further, the normal modes form a complete set in the sense that any function $f(y,z)$ satisfying the boundary conditions can be expressed in terms of the normal modes as follows:

$$f(y,z) = \sum_{n=1}^{\infty} a_n W_n(y,z). \quad (18)$$

The constants a_n's can be obtained using the orthogonality relation as

$$a_n = \frac{\int f(y,z)W_n(y,z)\,\mathrm{d}y\,\mathrm{d}z}{\int W_n^2\,\mathrm{d}y\,\mathrm{d}z}. \quad (19)$$

Now returning to the present flow problem, the fluid flow profile can be obtained in two steps. In the first step, the eigenmodes for the given cross section are calculated using the membrane analogy. In the second step, the true flow profile is calculated as summation of the normal modes whose contributions are obtained from the flow equation. Two examples are given below.

Example 4. Consider a rectangular channel with dimensions of $2a$ and $2b$ along y- and z-axes, respectively. Taking the origin at the center of cross section the normal modes of the rectangular membrane can easily be shown to be the following [11]:

$$W_{m,n}(y,z) = \cos\left(\frac{2m+1}{2a}\right)\pi y \cos\left(\frac{2n+1}{2b}\right)\pi z \quad (20)$$

and the corresponding natural frequency is

$$\omega_{m,n} = \sqrt{\left(\frac{2m+1}{2a}\right)^2 + \left(\frac{2n+1}{2b}\right)^2}\,\pi. \quad (21)$$

The flow profile in the rectangular channel can be written as

$$u(y,z) = \sum_{m,n=1}^{\infty} a_{m,n} W_{m,n}(y,z). \quad (22)$$

Substituting Eq. (22) into Eq. (1) one gets

$$\sum_{m,n=1}^{\infty} a_{m,n}\left(\frac{\partial^2 W_{m,n}}{\partial y^2} + \frac{\partial^2 W_{m,n}}{\partial z^2}\right) = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}. \quad (23)$$

The above expression can further be simplified using Eq. (16) as

$$\sum_{m,n=1}^{\infty} a_{m,n}\left(-\omega_{m,n}^2 W_{m,n}\right) = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}. \quad (24)$$

The unknown constants are obtained using orthogonality relation (see Eq. (17)) as

$$a_{m,n} = -\left(\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right)\frac{\int W_{m,n}\,\mathrm{d}y\,\mathrm{d}z}{\omega_{m,n}^2 \int W_{m,n}^2\,\mathrm{d}y\,\mathrm{d}z} \quad (25)$$

or finally as

$$a_{m,n} = -\frac{16}{(2m+1)(2n+1)\pi^2\omega_{m,n}^2}\left(\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right)\sin\left(\frac{2m+1}{2}\right)$$
$$\times\,\pi\sin\left(\frac{2n+1}{2}\right)\pi \quad (26)$$

or as

$$a_{m,n} = -\frac{16}{(2m+1)(2n+1)\pi^2\omega_{m,n}^2}\left(\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right)(-1)^m(-1)^n. \quad (27)$$

The same flow problem has been solved in more direct manner [6] and the solution is given in the following form:

$$u(y,z) = -\frac{16a^2}{\pi^3}\left(\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right)\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} \times (-1)^n\left[1-\frac{\cosh\left(\frac{2n+1}{2a}\right)\pi z}{\cosh\left(\frac{2n+1}{2a}\right)\pi b}\right]\cos\left(\frac{2n+1}{2a}\right)\pi y. \tag{28}$$

Since,

$$\left[1-\frac{\cosh\left(\frac{2n+1}{2a}\right)\pi z}{\cosh\left(\frac{2n+1}{2a}\right)\pi b}\right] = \sum_{m=0}^{\infty}\alpha_m\cos\left(\frac{2m+1}{2b}\right)\pi z, \tag{29}$$

where $\alpha_m's$ can be easily found out as

$$\alpha_m = (-1)^m\frac{2}{s}\frac{(r/a)^2}{(r/a)^2+(s/b)^2} \tag{30}$$

with $r = \frac{2n+1}{2}\pi$ and $s = \frac{2m+1}{2}\pi$, the expression (28) can be written in the form given by Eq. (27). One may notice that the solution given by mode superposition method is more symmetric than that found in literature.

Example 5. Consider flow through a channel of semi-circular cross section with radius a. The corresponding membrane problem can be solved in polar co-ordinate system. The eigenvalue problem can be written as

$$-\omega^2 W(r,\theta) = \frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial^2 W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}. \tag{31}$$

For semi-circle we substitute $W(r,\theta) = R(r)\cos(2n+1)\theta$, (the sinusoidal terms are not included as the flow is symmetric about $\theta = 0$) that yields the following equation

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}+\left(\omega^2-\frac{(2n+1)^2}{r^2}\right)R = 0, \tag{32}$$

where the angle is measured from the plane of symmetry. Define a new variable $\tilde{r} = \omega r$. The above equation becomes

$$\tilde{r}^2\frac{\mathrm{d}^2R}{\mathrm{d}\tilde{r}^2}+\tilde{r}\frac{\mathrm{d}R}{\mathrm{d}\tilde{r}}+(\tilde{r}^2-(2n+1)^2)R = 0. \tag{33}$$

The solution can be written in the following form

$$R(\tilde{r}) = C_1J_{2n+1}(\tilde{r})+C_2Y_{2n+1}(\tilde{r}), \tag{34}$$

where J_{2n+1} and Y_{2n+1} are the Bessel's function of the first and second kind, respectively. Since the value of R vanishes at the origin one has $C_2 = 0$ (since $Y_{2n+1}(0)\to\infty$). Other boundary condition $R(a\omega) = 0$ is satisfied if

$$J_{2n+1}(a\omega) = 0. \tag{35}$$

The solution of Eq. (35) can be obtained numerically. It is known that for this problem countable infinite solutions (say, λ_1, λ_2, ... λ_m,...) exist so that

$$\omega_{m,n} = \frac{\lambda_m}{a} \quad (m = 1,2,3\ldots). \tag{36}$$

Thus the mode shapes of the semicircular membrane are given as

$$(W_{m,n}(r,\theta) = J_{2n+1}(\omega_{m,n}r)\cos(2n+1)\theta, \quad (m = 1,2,3\ldots,\ n = 1,2,3\ldots). \tag{37}$$

The modes are orthogonal, that is,

$$\int W_{m,n}(r,\theta)W_{k,l}(r,\theta)r\,\mathrm{d}r\,\mathrm{d}\theta = 0, \quad m\neq k,\ n\neq l. \tag{38}$$

This can be easily proved because the cosine functions are orthogonal and the Bessel's functions have the following property [12]

$$\int_0^a J_n(\omega_{mn}r)J_n(\omega_{kn}r)r\,\mathrm{d}r\,\mathrm{d}\theta = 0 \quad m\neq k.$$

The original flow problem can be stated in polar co-ordinates as

$$\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial^2 W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2} = \frac{1}{\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right). \tag{39}$$

Assume $W(r,\theta) = \sum_{m,n=1}^{\infty}a_{mn}W_{m,n}(r,\theta)$. From the above analysis, using Eq. (31), one gets the following equation

$$\frac{1}{\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) = -\sum_{m,n=1}^{\infty}a_{mn}\omega_{mn}^2W_{m,n}(r,\theta) \tag{40}$$

which after applying the orthogonality relations yields,

$$a_{mn} = -\frac{1}{\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\frac{1}{\omega_{mn}^2}\frac{\int W_{m,n}r\,\mathrm{d}r\,\mathrm{d}\theta}{\int W_{m,n}^2r\,\mathrm{d}r\,\mathrm{d}\theta}. \tag{41}$$

This completes the solution of the problem. It may be noted that the same method can be used for a cross section in the form of a sector of a circle with sector angle 2α. In this case, the assumed solution is $W(r,\theta) = R(r)\cos\frac{(2n+1)\pi\theta}{2\alpha}$ and the solution involves Bessel functions of order $v = \frac{(2n+1)\pi}{2\alpha}$.

5. Variational method

The analytical closed form and infinite series form of solution of a given flow problem may not be possible if the channel cross section is very complicated. In those cases, approximate solutions are to be sought. In this section, a variational formulation of the flow problem is given that can be used to obtain approximate flow profile for any channel. It may be seen that velocity profile is obtained by minimizing the following functional:

$$J[u] = \oint\int\left\{\frac{\mu}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]+u\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\right\}\mathrm{d}A, \tag{42}$$

where $u(y,z)$ is a function that satisfies the essential boundary condition, i.e., $u(y,z) = 0$ at the boundary. The functional attains optimal value when Eq. (1) is satisfied. In fact the condition of optimality for any function that satisfies the boundary condition is

$$\delta J[u] = -\int \left\{ \mu \left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] - \left(\frac{dp}{dx} \right) \right\} \delta u \, da = 0, \tag{43}$$

for all possible variation δu. That occurs if and only if

$$\mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \frac{dp}{dx}. \tag{44}$$

Many problems can be solved approximately by choosing appropriate trial functions $u(y,z)$ that satisfy the boundary condition and minimizing the functional (42). Two examples are given below.

Example 6. Consider the problem of flow through a channel of elliptical cross section whose boundary is given as $\varphi(y,z) = \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0$. Let the trial function be $u(y,z) = m\left(\frac{y^2}{a^2} + \frac{z^2}{b^2} - 1\right)$. The unknown quantity m can be obtained if $J[u]$ is minimized with respect to m. Clearly,

$$J[u] = \int \left[2\mu m^2 \left(\frac{y^2}{a^4} + \frac{z^2}{b^4} \right) + m \frac{dp}{dx} \left(\frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 \right) \right] dy \, dz, \tag{45}$$

$$= \frac{\mu m^2}{2} \left(\frac{b}{a} + \frac{a}{b} \right) \pi + \left(\frac{dp}{dx} \right) m \cdot \left(-\frac{\pi ab}{2} \right). \tag{46}$$

The value of m is obtained by minimizing J. Hence, $\frac{dJ}{dm} = 0 \Rightarrow m = \frac{1}{\mu} \frac{dp}{dx} \frac{a^2 b^2}{2(a^2+b^2)}$. Thus the flow profile becomes

$$u(y,z) = \frac{a^2 b^2}{2(a^2 + b^2)} \left(\frac{1}{\mu} \frac{dp}{dx} \right) \left(\frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 \right). \tag{47}$$

In this case, the result obtained by approximate method is also exact because the trial function satisfies the boundary condition as well as the governing equation.

Example 7. Consider a channel whose boundary is defined in polar co-ordinates as $r = a(1 + \varepsilon \cos n\theta)$. To obtain the approximate flow profile for this problem the variational functional (42) is formulated in terms of polar co-ordinates as

$$J[u] = \oint \int \left\{ \frac{\mu}{2} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2 \right] + u \left(\frac{dp}{dx} \right) \right\} r \, dr \, d\theta. \tag{48}$$

The approximate solution can be taken as

$$u = m(r + a) \left(\frac{r}{1 + \varepsilon \cos n\theta} - a \right), \tag{49}$$

where from the unknown constant can be calculated by substituting Eq. (49) into Eq. (48) and minimizing the integral with respect to m. This gives

$$m = \frac{1}{\mu} \left(\frac{dp}{dx} \right) \frac{C_1}{C_2}, \tag{50}$$

where

$$C_1 = \frac{1}{12} \int_0^{2\pi} \left[(1 + \varepsilon \cos n\theta)^3 + 2(1 + \varepsilon \cos n\theta)^2 \right] d\theta \tag{51}$$

and

$$\begin{aligned} C_2 = & \int_0^{2\pi} \left[\frac{1}{6}(1 + \varepsilon \cos n\theta)^2 + \frac{1}{3}(1 + \varepsilon \cos n\theta) + \frac{1}{2} \right] d\theta \\ & + \int_0^{2\pi} \left[n^2 \varepsilon^2 \sin^2 n\theta \left(\frac{1}{4}(1 + \varepsilon \cos n\theta)^2 \right. \right. \\ & \left. \left. + \frac{2}{3}(1 + \varepsilon \cos n\theta) + \frac{1}{2} \right) \right] d\theta. \end{aligned} \tag{52}$$

In order to improve the accuracy of the result with variational method, the assumed flow profile can be approximated in a series form as

$$u = \sum_{k=1}^{\infty} m_k \phi_k(y,z), \tag{53}$$

where the trial functions $\phi_k(y,z)$ vanish at the boundaries. The unknown constants, m_k's are obtained by substituting Eq. (53) in Eq. (48) and minimizing the functional using $\frac{\partial J(m_1, m_2, \ldots)}{\partial m_i} = 0$, $(i = 1, 2, \ldots)$. For example, the trial function in Example 7 can be written in series form as

$$u = (r + a) \left(\frac{r}{1 + \varepsilon \cos n\theta} - a \right) \sum_{k,l=0}^{\infty} m_{kl} r^k \cos l\theta,$$

where the unknown constants are obtained by minimizing the integral (48).

6. Conclusions

Steady viscous flow within a straight microchannel of arbitrary cross section is analyzed. Three methods have been discussed to obtain exact and approximate flow profile. The first method uses complex functional analysis whereas the second method exploits analogy with the problem of transverse vibration of a membrane. In the third method a variational formulation is given whereby approximate flow velocity is easily obtained using optimization technique.

References

[1] N.-T. Nguyen, S. Werely, Fundamentals and Applications of Microfluidics, Artech House Publishers, 2006.
[2] J. Barrat, L. Bocquet, Large slip effect at a non-wetting fluid solid interface, Physical Review Letters 82 (1999) 4671–4674.
[3] D. Tretheway, C. Meinhart, Apparent fluid slip at hydrophobic microchannel walls, Physics of Fluids 14 (2002) L9–L12.
[4] K. Watanabe, Yanuar, H. Mizunuma, Slip of Newtonian fluids at solid boundary, JSME International Journal Series B 41 (1998) 525–529.
[5] R.W. Fox, A.T. McDonald, Introduction to Fluid Mechanics, John Wiley and Sons Inc., NY, 2001.

[6] D.B. Holmes, J.R. Vermeulen, Velocity profiles in ducts with rectangular cross-sections, Chemical Engineering Science 23 (1968) 717–722.
[7] S.P. Timoshenko, J.N. Goodier, Theory of Elasticity, McGraw-Hill Book Company, 1970.
[8] Saint-Venant, Mémoirés savants strangers, 14, 1855.
[9] N.I. Muskhelishvili, Some Basic Problems of the Theory of Elasticity, Groningen, Netherlands, 1963.
[10] I.S. Sokolnikoff, Mathematical Theory of Elasticity, McGraw-Hill Book Company, NY, 1956.
[11] L. Meirovitch, Analytical Methods in Vibration, Macmillan Book Company, London, 1967.
[12] E. Kreyszig, Advanced Engineering Mathematics, John Wiley and Sons Inc., NY, 1999.